General Relativity | Week 13

Last time:

~~[scribbled]~~ Local well-posedness theorem for quasilinear wave equations:

For any fixed $k > 2n+2$, the in. value problem

$$\begin{cases} \partial_\alpha (G^{\alpha\beta}(x,\psi)\, \partial_\beta \psi) = N(x,\psi,\partial\psi) \\ \psi|_{t=0} = \psi_0, \quad \partial_t \psi|_{t=0} = \psi_1 \end{cases} \quad ①$$

with $|G^{\alpha\beta} - \eta^{\alpha\beta}| < \frac{1}{10 \cdot n}$, satisfies the following:

if $\|\psi_0\|_{H^k} + \|\psi_1\|_{H^{k-1}} = M$, $\exists\, T = T(M) > 0$ and a unique solution ψ to ①, which depends continuously on the initial data, in the sense that

if $\|\psi_0^{(n)} - \psi_0\|_{H^k} + \|\psi_1^{(n)} - \psi_1\|_{H^{k-1}} \xrightarrow{n\to\infty} 0$

Then $\limsup_n T^{(n)} \geq T$ (times of existence)

Cauchy stability $\longrightarrow$ and $\|\psi^{(n)} - \psi\|_{L^\infty H^k} + \|\partial_t \psi^{(n)} - \partial_t \psi\|_{L^\infty H^{k-1}} \xrightarrow{n} 0$.

Special case: Cauchy stability of $\psi = 0$ (when $N(x,0,0) = 0$)

By gluing together local solutions:

Corollary:

Let $\Sigma^n \subseteq M^{n+1}$ and consider the IVP

$$\begin{cases} \Box_{g(x,\psi)} \psi = N(x,\psi,d\psi) \\ \psi|_\Sigma = \psi_0, \quad \hat{n}(\psi)|_\Sigma = \psi_1. \end{cases}$$

Where $\hat{n}$ unit normal to Σ and $g(x,\psi)$ is Lorentzian, Σ spacelike with respect to $g(x,\psi_0)$.

and $(\psi_0, \psi_1) \in C^\infty(\Sigma) \times C^\infty(\Sigma)$,

$N : M \times \mathbb{R} \times T^*M \to \mathbb{R}$ smooth, $g(x, \cdot) : \mathbb{R} \to T_xM \otimes T_xM$.

Then $\exists\, U \supseteq \Sigma$ open and unique solution ψ on U.

The vacuum Einstein equations

$Ric - \frac{1}{2} R \cdot g = 0 \iff Ric = 0$ $(n > 2)$. We would like to construct solutions

From the expression $R^a{}_{\beta\gamma\delta} = \partial_\gamma \Gamma^a_{\beta\delta} - \partial_\delta \Gamma^a_{\beta\gamma} + \Gamma \cdot \Gamma$,

we compute

$$Ric_{\mu\nu} = -\frac{1}{2} g^{\alpha\beta} \left(\partial_\alpha \partial_\beta g_{\mu\nu} + \partial_\mu \partial_\nu g_{\alpha\beta} - \partial_\alpha \partial_\nu g_{\mu\beta} - \partial_\mu \partial_\beta g_{\alpha\nu} \right) + N_{\mu\nu}(g, \partial g)$$

If it was just the first term, then

$Ric = 0$ would be of the form $\Box_g g_{\mu\nu} = \tilde{N}_{\mu\nu}(g, \partial g)$,

so we could solve a regular initial value problem.

But this is not true in a general coordinate system (covariance of the equations: non uniqueness)

Can we find good coordinates in which only the first term survives?

Wave coordinates:

$\Box_g x^\mu = 0 \iff g^{\alpha\beta} \Gamma^\mu_{\alpha\beta} = 0$ $\rightarrow$ Not an 1-form: it is coordinate dependent

Setting $\partial_\mu := g_{\mu\nu} g^{\alpha\beta} \Gamma^\nu_{\alpha\beta} = g^{\alpha\beta} \left(\partial_\alpha g_{\mu\beta} - \frac{1}{2} \partial_\mu g_{\alpha\beta} \right)$

Then $\Box_g x^\mu = 0 \quad \forall \mu = 0,1,2,\dots,n \iff \Lambda_\mu = 0 \quad \forall \mu = 0,1,\dots,n.$

Note:

$$Ric_{\mu\nu} = -\frac{1}{2} g^{\alpha\beta} \partial_\alpha \partial_\beta g_{\mu\nu} + \frac{1}{2}\left(\partial_\mu \Lambda_\nu + \partial_\nu \Lambda_\mu\right) + N(g, \partial g).$$

Def: Reduced Ricci "tensor":

$$\widetilde{Ric}_{\mu\nu} = Ric_{\mu\nu} - \frac{1}{2}\left(\partial_\mu \Lambda_\nu + \partial_\nu \Lambda_\mu\right)$$

So: If $\widetilde{Ric}_{\mu\nu} = 0$ <u>and</u> $\Lambda_\mu = 0 \implies Ric_{\mu\nu} = 0$

Do we have a good equation for Λ? In general no, but:

<u>Lemma</u>: If $\widetilde{Ric} = 0$ then $\Box_g \Lambda_\mu + A_\mu^{\alpha\beta} \partial_\alpha \Lambda_\beta + B_\mu^a \Lambda_a = 0$

<u>Proof</u>:

$\widetilde{Ric} = 0 \iff$

$$\iff Ric_{\mu\nu} = \frac{1}{2}\left(\partial_\mu \Lambda_\nu + \partial_\nu \Lambda_\mu\right) = \frac{1}{2}\left(\nabla_\mu \Lambda_\nu + \nabla_\nu \Lambda_\mu\right) + \Gamma \cdot \Lambda$$

So $R = g^{\mu\nu} Ric_{\mu\nu} = \nabla^a \Lambda_a + \Gamma \cdot \Lambda$

From 2^{nd} Bianchi identity:

$$\nabla^a Ric_{a\mu} - \frac{1}{2} \nabla_\mu R = 0$$

$$\implies \frac{1}{2} \nabla^a \nabla_a \Lambda_\mu + \underbrace{\frac{1}{2} \nabla^a \nabla_\mu \Lambda_a - \frac{1}{2} \nabla_\mu \nabla^a \Lambda_a}_{:= R \cdot \Lambda} + \Gamma \cdot \Lambda = 0$$

∎

So: A strategy to construct solutions:

Solve $\widetilde{Ric}_{\mu\nu} = 0$ (of the form $\Box_g g_{\mu\nu} = N_{\mu\nu}(g, \partial g)$)

with initial data

$$\left.\begin{array}{l} g_{\mu\nu}|_{t=0}, \quad \partial_t g_{\mu\nu}|_{t=0} \\ \text{and} \quad \partial_\nu|_{t=0} = 0, \quad \partial_t \partial_\nu|_{t=0} = 0 \end{array}\right\} \text{overdetermined!}$$

Is there a way to "untangle" the overdetermined nature?

Let us introduce a more geometric viewpoint.

If (M, g) is a Lorentzian manifold satisfying $Ric_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu}$ and $\Sigma \subset M$ is spacelike, then the induced Riem. metric $\bar{g}$ and <u>second fundamental</u> form k (with respect to a unit normal $\hat{n}$: $k(X,Y) = g(\nabla_X \hat{n}, Y)$) satisfy the Gauss-Codazzi eqns; from which we obtain the <u>constraint equations</u>:

$$① \begin{cases} \bar{R} + \underbrace{(tr_{\bar{g}} k)^2 - \|k\|^2_{\bar{g}}}_{-\|k \wedge k\|^2_{\bar{g}}} = 16\pi\, T(\hat{n}, \hat{n}) \longrightarrow \text{Hamiltonian Constraint} \\ \underbrace{div_{\bar{g}} (k - tr_{\bar{g}} k \cdot \bar{g})}_{\bar{\nabla}^i k_{ij} - \bar{\nabla}_j (tr_{\bar{g}} k)} = 8\pi\, T(\hat{n}, \cdot) \longrightarrow \text{Momentum Constraint} \end{cases}$$

$(n+1)$ equations.

So: Initial data for the vacuum Einstein equation:

$(\Sigma, \bar{g}, k)$ satisfying ① with $T_{\mu\nu} = 0$.

Note: ① is underdetermined elliptic. For instance, if $\hat{g}$ is any Riem. metric on Σ and we want to find "time-symmetric" (i.e. $k=0$) initial data of the form

$$(g, k) = (\phi^4 \hat{g}, 0) \quad (\text{for } n=3)$$

then ① is equivalent to the following elliptic equation for ϕ:
$$\Delta_{\hat{g}} \phi = \frac{1}{8} \hat{R} \phi.$$

Def: Let $(\Sigma, \bar{g}, k)$ be an (vacuum) initial data set.

symmetric (0,2)-form

n-manifold Riem. metric

A development (M, g) is a globally hyperbolic spacetime solving $Ric = 0$, together with an embedding $i: \Sigma \longrightarrow M$ such that $i(\Sigma)$ is a Cauchy hypersurface and the induced metric/second fundamental form are $(\bar{g}, k)$.

Theorem (Choquet-Bruhat, 1952)

$\forall$ smooth initial data set $(\mathbb{R}^n, \bar{g}, k)$, with $\|\bar{g}-\bar{g}_E\|_{H^{2n+2}} + \|k\|_{H^{2n+1}} < +\infty$

$\exists$ development $((-\delta,\delta)\times\mathbb{R}^n, g)$ which is geometrically unique.

Sketch of proof:

We want to end up solving $\widetilde{Ric}=0 \Leftrightarrow \Box_g g_{\mu\nu} = Q_{\mu\nu}(g,\partial g)$.

So we need to choose initial data such that $\Gamma_\mu|_{t=0}=0$, $\partial_t \Gamma_\mu|_{t=0}=0$ (so $\Gamma\equiv 0$)

At $t=0$: Set $g_{00}=-1$, $g_{0i}=0$ (so that $\partial_t = \hat{n}$)

then $g|_{t=0} = \left[\begin{array}{c|ccc} -1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & \bar{g} & \\ 0 & & & \end{array}\right]$, $\partial_t g|_{t=0} = \left[\begin{array}{c|c} * & * \cdots \\ \hline * & 2k \\ * & \end{array}\right]$

(So $\partial_t g_{ij}|_{t=0} = 2k_{ij}$; this is because

$$\partial_t g_{ij} = \partial_t \langle \partial_i, \partial_j\rangle = \langle \nabla_{\partial_t}\partial_i, \partial_j\rangle + \langle \partial_i, \nabla_{\partial_t}\partial_j\rangle$$
$$= \langle \nabla_{\partial_i} \underset{\hat{n}}{\underset{\shortparallel}{\partial_t}}, \partial_j\rangle + \langle \partial_i, \nabla_{\partial_j}\partial_t\rangle$$
$$= k_{ij} + k_{ji} \quad)$$

So $g_{ij}|_{t=0}$, $\partial_t g_{ij}|_{t=0}$, $i,j=1,\dots,n$ are determined by $(\bar{g},k)$

We still have to choose $\partial_t g_{0a}|_{t=0}$, $a=0,\dots,n$.

The requirement that $\Gamma_\mu|_{t=0} \Leftarrow g^{a\beta}(2\partial_a g_{\mu\beta} - \partial_\mu g_{a\beta})=0$, $\mu=0,\dots,n$

fixes all the components $\partial_t g_{0a}|_{t=0}$

There is no more freedom left! How do we ensure that $\partial_t \partial_\nu|_{t=0} = 0$?

This is magically implied by the constraint equations!

Since we are solving $\widetilde{Ric} = 0 \Leftrightarrow Ric_{\mu\nu} = \frac{1}{2}(\partial_\mu \partial_\nu + \partial_\nu \partial_\mu)$

The constraint equations:

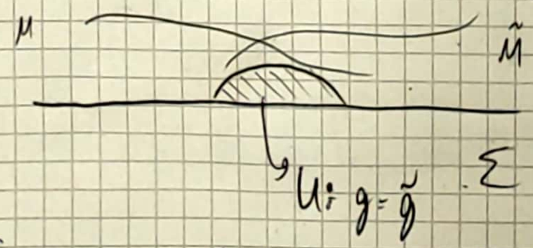
$$\left.\begin{array}{l} Ric_{00} - \frac{1}{2} R g_{00}\big|_{t=0} = 0 \\ Ric_{0i}\big|_{t=0} = 0 \end{array}\right\} \begin{array}{l} \partial_0 \partial_0 + \frac{1}{2} \partial^\mu \partial_\mu\big|_{t=0} = 0 \\ \partial_0 \partial_i\big|_{t=0} = 0 \end{array}$$

$$\Rightarrow \partial_0 \partial_a\big|_{t=0} = 0 \quad \forall a = 0, \dots, n$$

So: We get existence by the local well-posedness theory for non-linear wave equations.

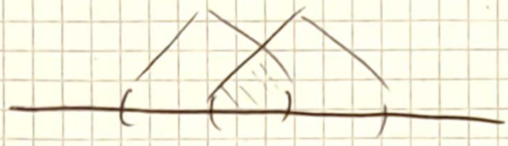
Uniqueness: Suppose (M, g) and $(\tilde{M}, \tilde{g})$ are two developments.

In wave coordinates (with $x^i|_{t=0}$ fixed so that the embeddings i and $\tilde{i}$ agree, and $\partial_t|_{t=0} = \hat{n}$)



So we can "glue" the developments.

If the initial data are "large":



Restrict in small balls, construct the solution in the corresponding domain of development, glue along overlaps.

Theorem (Choquet-Bruhat, Geroch '69)

Given a smooth initial data set $(\Sigma, \bar{g}, k)$, there exists a unique, maximal globally hyperbolic development (M, g)

↳ i.e.: If $(\tilde{M}, \tilde{g})$ is any other development with corresponding embedding $\tilde{i}: \Sigma \to \tilde{M}$

then $\exists$ isometric embedding $F: \tilde{M} \to M$ with $F \circ \tilde{i} = i$.

Philosophically: General relativity is a deterministic theory

Initial data uniquely determine the future and the past.

Where does the spacetime end?

Is (M, g) geodesically complete if this is the case for $(\Sigma, \bar{g}, k)$?

Answer: No! Example: Schwarzschild ($r = 0$ at finite affine distance)

Next time: Some criteria for incompleteness